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A Lagrangian interpolation method for three-point problems

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Abstract—Three-point problems, traditionally solved using graphical constructions, can be formulated as Lagrangian interpolation problems and solved numerically using standard methods. In addition to calculating strike and dip of a planar geologic feature, three-point interpolation gives elevation of the feature at an arbitrary point with minimum effort.

INTRODUCTION

GRAPHICAL solution of three-point problems is typically taught as part of elementary structural geology (e.g. Billings 1972, Ragan 1985, Suppe 1985, Rowland 1986, Marshak & Mitra 1988), groundwater hydrology (Freeze & Cherry 1979) or engineering geology (Rahn 1986). Three-point problems, however, can also be formulated as Lagrangian interpolation problems and solved using standard techniques for systems of linear equations. Calculating strike and dip is no more difficult using interpolation methods than using graphical methods, and calculating the elevation of a dipping plane at an arbitrary point becomes a straightforward task. Although three-point interpolation plays an important role in finite element solutions of partial differential equations, where linear shape functions can be used to interpolate values in triangular elements (e.g. Zienkiewicz 1977), the method does not appear to be widely known among geologists.

The classic interpolation problem is to find Nth degree polynomial values that pass exactly through $N+1$ known points in a plane. Strike, dip and elevation of a geologic feature can be calculated through the analogous procedure of finding values of a plane that passes exactly through three known points in space. The adjective *Lagrangian* describes interpolation methods that express these explicitly in terms of the co-ordinates of known points (Hildebrand 1987), e.g. borehole or outcrop locations.

This note presents Lagrangian interpolation methods for the solution of two geologically separate but mathematically similar problems: calculation of the depth or elevation of a dipping plane at a given point, and calculation of strike and dip of a dipping plane. The first problem arises frequently in drilling programs, when a geologist may need to order materials or estimate drilling times based upon previously-collected data. In both problems, the positive x-direction corresponds to east, the positive y-direction corresponds to north and the z-axis is positive upwards.

ELEVATION OF A DIPPING PLANE

Assume that x, y, z co-ordinates of a planar geologic feature are known at three points, perhaps from boreholes or outcrops, and the goal is to calculate z at a fourth point for which only x and y are known. The general formula for a plane is

$$z = a_0 + a_1 x + a_2 y. \tag{1}$$

Thus, the three known points and one unknown point can be represented by the system of four linear equations

$$\begin{bmatrix} 1 & x_1 & y_1 & 0 \\ 1 & x_2 & y_2 & 0 \\ 1 & x_3 & y_3 & 0 \\ 1 & x_4 & y_4 & -1 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ z_4 \end{Bmatrix} = \begin{Bmatrix} z_1 \\ z_2 \\ z_3 \\ 0 \end{Bmatrix}. \tag{2}$$

Equations (2) can be solved for a_0, a_1, a_2 and z_4 using any number of standard methods, including mathematical subroutine libraries, equation-solving programs or hand-held calculators with matrix algebra capabilities. An alternative that at first seems attractive is to solve the set of equations

$$\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} z_1 \\ z_2 \\ z_3 \end{Bmatrix} \tag{3}$$

for a_0, a_1 and a_2, and then substitute these values into equation (1) to calculate z_4. This approach should be avoided for two reasons. First, the Vandermonde coefficient matrix in equations (3) is commonly ill-conditioned, producing values of z_4 that will be less accurate than simultaneous solution for a_0, a_1, a_2 and z_4 (Press *et al.* 1989, pp. 92–95). Second, if a computer or calculator with matrix capabilities is used, then it is no more difficult to solve a set of four equations than it is to solve a set of three equations. Therefore, one might as well solve directly for z_4 if this is the desired quantity.

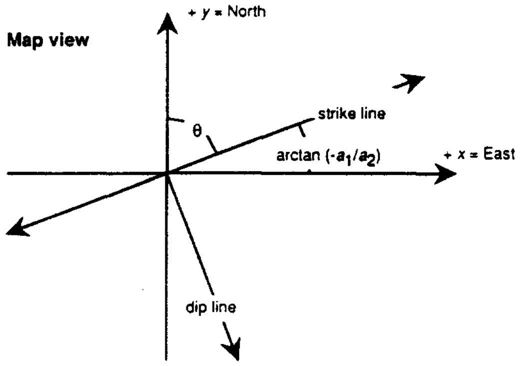

Fig. 1. Illustration of variables and co-ordinates used in the three-point interpolation problem. The strike line is defined as a line of equal elevation dz/dx = dz/dy = 0, and the tangent of true dip is gradz in the strike line–dip line co-ordinate system.

STRIKE AND DIP OF A PLANE

Calculation of strike and dip requires some additional work. The strike of a dipping layer is defined by the intersection of that layer with an imaginary horizontal plane, forming a line of equal elevation. Thus strike can be calculated by setting the total derivative of equation (1) equal to zero

$$\frac{\mathrm{d}}{\mathrm{d}x}(a_0 + a_1x + a_2y) = a_1 + a_2\frac{\partial y}{\partial x} = 0 \tag{4}$$

and solving for $\partial y/\partial x = -a_1/a_2$, which is the tangent of the angle between the positive x-axis and the strike line. Taking the total derivative of equation (1) with respect to y will produce the same result. The strike angle (i.e. the angle measured clockwise from the positive y-axis to the strike line) is given by (Fig. 1)

$$\theta = 90° - \arctan(-a_1/a_2). \tag{5}$$

The tangent of the dip angle is calculated by taking the gradient of equation (1)

$$\operatorname{grad} z = \frac{\partial z}{\partial x}\hat{\mathbf{x}} + \frac{\partial z}{\partial y}\hat{\mathbf{y}} = a_1\hat{\mathbf{x}} + a_2\hat{\mathbf{y}}, \tag{6}$$

where $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are, respectively, unit vectors in the positive x- and y-directions. The direction of true dip will in all cases be perpendicular to strike, so equation (6) must be re-written to yield the gradient perpendicular to strike by rotating the x–y orthogonal co-ordinate system to coincide with the orthogonal strike and dip lines. Therefore the expression for dip angle (i.e. the maximum angle between an imaginary horizontal plane and the dipping plane) is

$$\delta = \arctan[a_1 \sin(90° \pm \theta) + a_2 \cos(90° \pm \theta)]. \tag{7}$$

Inspection of (7) shows that the term containing a_1 is the gradient along the strike line and that the term containing a_2 is the gradient along the dip line. In theory, then, $a_1 = 0$ in order for the strike line to be a line of equal elevation. As will be shown in the example problem below, however, calculated values of a_1 are typically small, non-zero values. Vacher (1989) also uses a gradient method, but one which does not involve interpolation, to calculate the magnitude of dip using the formula

$$\tan\delta = |\operatorname{grad} z| = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}, \tag{8}$$

which is equivalent to (7) in this note. Vacher's expression for strike angle is also identical to (5), save for the 90° difference between north and x co-ordinate axes.

The azimuth of the dip line is given by 90° ± θ, where addition or subtraction of θ is chosen in order to get $\delta < 0$. If only strike and dip are to be calculated, then equations (3) may be used to calculate a_0, a_1 and a_2, *or* dummy values for x_4 and y_4 may be used in equations (2).

EXAMPLE PROBLEM

To illustrate an application of the interpolation method, consider a plane passing through known (x,y,z) co-ordinates (7,6,−8), (12,0,−10) and (2,−5,−12). The goal is to calculate (a) elevation of the plane at $x = 7$ $y = 2$, and (b) strike and dip of the plane. Substitution of the known values into equations (2) gives

$$\begin{bmatrix} 1 & 7 & 6 & 0 \\ 1 & 12 & 0 & 0 \\ 1 & 2 & -5 & 0 \\ 1 & 7 & 2 & -1 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ z_4 \end{Bmatrix} = \begin{Bmatrix} -8 \\ -10 \\ -12 \\ 0 \end{Bmatrix} \tag{9}$$

which is solved to yield, to two decimal places,

$$\begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ z_4 \end{Bmatrix} = \begin{Bmatrix} -10.28 \\ 0.02 \\ 0.35 \\ -9.41 \end{Bmatrix}. \tag{10}$$

This example problem was solved twice, using both the computer program *Mathematica* (Wolfram 1988) and a Hewlett-Packard HP-15C calculator. From equation (5), the strike angle is given by

$$\theta = 90° - \arctan(-0.02/0.35) = 93.27° \tag{11}$$

or, using the geologic convention that strike must range from 270° through 360° to 090°, 273.27°. This problem is systematically resolved by adding 180° if 90° ≤ θ ≤ 180° and subtracting 180° if 180° ≤ θ ≤ 270°. From equation (7) the dip angle is

$$\delta = \arctan[0.02 \sin(90° + 93.27°) + 0.35 \cos(90° + 93.27°)] = -19.31° \tag{12}$$

along an azimuth of 90° + 93.27° = 183.27°. The reader can verify these strike and dip values using traditional methods. Calculated apparent dip along the strike line, which must in theory equal zero, is in this case

$$\arctan[0.02 \sin(90° + 93.27°)] = -0.06° \tag{13}$$

which is negligible.

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